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On the stability of periodic orbits of high dimensional autonomous Hamiltonian systems

Ch. Skokos [a,b]

[a] *Research Center for Astronomy, Academy of Athens, 14 Anagnostopoulou Street, GR-10673 Athens, Greece*
[b] *Department of Physics, University of Athens, Panepistimiopolis, GR-15784 Zografos, Greece*

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This work is dedicated to the memory of Dr. C. Polymilis who suggested the subject of the present paper; unfortunately he passed away before this work was completed.

Abstract

We study the stability of periodic orbits of autonomous Hamiltonian systems with $N+1$ degrees of freedom or equivalently of $2N$-dimensional symplectic maps, with $N \geq 1$. We classify the different stability types, introducing a new terminology which is perfectly suited for systems with many degrees of freedom, since it clearly reflects the configuration of the eigenvalues of the corresponding monodromy matrix, on the complex plane. The different stability types correspond to different regions of the N-dimensional parameter space $\mathcal{S}$, defined by the coefficients of the characteristic polynomial of the monodromy matrix. All the possible direct transitions between different stability types are classified, and the corresponding transition hypersurface in $\mathcal{S}$ is determined. The dimension of the transition hypersurface is an indicator of how probable to happen is the corresponding transition. As an application of the general results we consider the well-known cases of Hamiltonian systems with two and three degrees of freedom. We also describe in detail the different stability regions in the three-dimensional parameter space $\mathcal{S}$ of a Hamiltonian system with four degrees of freedom or equivalently of a six-dimensional symplectic map.

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1. Introduction

The stability of periodic orbits of high dimensional autonomous Hamiltonian systems (with more than three degrees of freedom) and equivalently of high dimensional symplectic maps is a significant problem in nonlinear dynamics having a variety of applications ranging from celestial mechanics (e.g. [1–4]) to chemical physics (e.g. [5–7]). The problem has been studied in the past by a number of authors, and important results have been given for systems with two and three degrees of freedom [8,9]. Due to the complexity of the problem it is inevitable to restrict attention to low dimensions. Nevertheless, some work has been also done on systems with many degrees of

E-mail address: hskokos@cc.uoa.gr (Ch. Skokos).

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freedom [10–12], but a general theory in dimensions higher than three is still lacking. Most of the activity so far was focused on the derivation of stability boundaries for symplectic maps [10,12]. Howard and MacKay [10], based on the observation that the introduction of the stability indices reduces the characteristic polynomial which gives the eigenvalues of the monodromy matrix to a polynomial with half the original order, succeeded in obtaining results for the stability boundaries of symplectic maps of dimension as high as eight.

The aim of the present paper is different. Since general results are difficult to get for the stability boundaries in dimensions higher than eight, we restrict attention in classifying all the possible stability types in the general case of a Hamiltonian system with $N+1$ degrees of freedom, which corresponds to a $2N$-dimensional symplectic map. We introduce a new terminology for all the possible stability types, which help us in studying the direct transitions between these types.

The properties of periodic orbits of Hamiltonian systems with three degrees of freedom or equivalently of four-dimensional symplectic maps, have been studied extensively (e.g. [13–24]). In such systems particular attention has been given to complex instability (e.g. [25–31]), a type of instability that does not appear in systems with two degrees of freedom. After having studied systems with three degrees of freedom, the next step towards understanding instabilities in multidimensional systems is the case of four degrees of freedom. A four degrees of freedom system is the simplest case where new different types of complex instability appear. So we study in detail the case of a Hamiltonian system with four degrees of freedom which corresponds to a six-dimensional symplectic map. We note that the stability parameter space of such systems is three-dimensional and can be visualized. The boundaries of all the stability types are obtained helping us defining the various transitions not only from stability to instability but also between different types of instabilities.

The paper is organized as follows. In Section 2 we review briefly the basic theory of stability of Hamiltonian systems, giving definitions of many concepts like the monodromy matrix and the stability indices and providing some basic formulae for these quantities. The results of the present paper are presented in Sections 3 and 4. In particular in Section 3.1 we define the different stability types that are possible in the general case and we introduce a new terminology for them. In Section 3.2 we study the case of $N+1$ degrees of freedom when N is even or odd, counting all the possible stability types. In Section 3.3 the direct transitions between different stability types are studied. In Section 4 we apply the results of the previous two sections in some particular cases. For completeness sake in Sections 4.1 and 4.2 we review the stability types that appear in Hamiltonian systems with two and three degrees of freedom, respectively, while the complete study of the four degrees of freedom case is done in Section 4.3. Finally in Section 5 we summarize our results.

2. A quick reminder of the stability theory of Hamiltonian systems with *N*+*1* degrees of freedom

One of the main motivations for studying the stability of periodic orbits of Hamiltonian systems is its great significance for the dynamical behavior of the system. It is well known that non-periodic orbits near a stable periodic orbit are ordered, i.e. their evolution in time is similar to the behavior of the periodic orbit, while the unstable periodic orbits introduce chaotic behavior in the system.

Let us consider an autonomous Hamiltonian system H_0, not necessarily integrable, with $N+1$ degrees of freedom, where N is an integer with $N \geq 1$, which is perturbed. Then its Hamiltonian function can be written as

$$H = H_0 + \epsilon H_1, \tag{1}$$

where ϵ is the perturbation parameter. The equations of motion for this system can be expressed in the form

$$\dot{\boldsymbol{x}} = -\boldsymbol{J} \cdot \nabla \boldsymbol{H} = -\boldsymbol{J} \cdot \nabla(\boldsymbol{H}_0 + \epsilon \boldsymbol{H}_1), \tag{2}$$

with $\boldsymbol{x} = (q_1, q_2, \dots, q_{N+1}, p_1, p_2, \dots, p_{N+1})'$ and

$$\nabla \boldsymbol{H} = \left(\frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial q_2}, \dots, \frac{\partial H}{\partial q_{N+1}}, \frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2}, \dots, \frac{\partial H}{\partial p_{N+1}} \right)',$$

where q_i, $i = 1, 2, \dots, N+1$ are the generalized coordinates and p_i, $i = 1, 2, \dots, N+1$ the conjugate momenta and prime ($'$) denotes the transpose matrix. The matrix $\boldsymbol{J}$ has the following block form:

$$\boldsymbol{J} = \begin{pmatrix} \mathbf{0}_n & -\boldsymbol{I}_n \\ \boldsymbol{I}_n & \mathbf{0}_n \end{pmatrix}, \tag{3}$$

where $\boldsymbol{I}_n$ is the $n \times n$ identity with $n = N + 1$ and $\mathbf{0}_n$ is the $n \times n$ matrix with all its elements equal to zero.

The linear stability of a periodic orbit of this system with period T is determined by the solution of the linearized equations

$$\boldsymbol{J} \cdot \dot{\boldsymbol{\xi}} = (\boldsymbol{P}_0 + \epsilon \boldsymbol{P}_1) \cdot \boldsymbol{\xi} = \boldsymbol{P} \cdot \boldsymbol{\xi}, \tag{4}$$

where $\boldsymbol{\xi}$ is a vector denoting the deviation from the given periodic orbit in the $(2N + 2)$-dimensional phase space of the system and represented by a $(2N + 2) \times 1$ matrix, $\boldsymbol{P} = \boldsymbol{P}_0 + \epsilon \boldsymbol{P}_1$ is the Hessian matrix of the Hamiltonian (1) calculated on the periodic orbit the stability of which we study. The elements of matrix $\boldsymbol{P}$

$$P_{ij} = \frac{\partial^2 H}{\partial x_i \partial x_j}, \quad i, j = 1, 2, \dots, 2N + 2 \tag{5}$$

are T-periodic functions of time since the RHS of Eq. (5) is calculated for the T-periodic orbit. Eqs. (4) are the so-called variational equations of the system for the particular periodic orbit. A $(2N+2) \times (2N+2)$ matrix whose individual columns consist of $2N + 2$ linearly independent solutions of Eqs. (4) is called a fundamental matrix of solutions of the variational equations (4). The fundamental matrix $\boldsymbol{X}(t)$ whose solutions correspond to the initial conditions

$$\boldsymbol{X}(0) = \boldsymbol{I}_{2N+2}, \tag{6}$$

gives the evolution of the deviation vector $\boldsymbol{\xi}$ for $t = \kappa T$, $\kappa \in \mathbb{N}^*$ through the relation

$$\boldsymbol{\xi}(\kappa T) = [\boldsymbol{X}(T)]^{\kappa} \cdot \boldsymbol{\xi}(0). \tag{7}$$

The matrix

$$\boldsymbol{A} = \boldsymbol{X}(T) \tag{8}$$

is called the monodromy matrix and satisfies the symplectic condition [32]

$$\boldsymbol{A}' \cdot \boldsymbol{J} \cdot \boldsymbol{A} = \boldsymbol{J}. \tag{9}$$

The stability type of the periodic orbit is determined by knowing the nature of the eigenvalues of the matrix $\boldsymbol{A}$. Due to the symplectic condition (9) and due to the fact that the matrix coefficients are real, the eigenvalues of matrix $\boldsymbol{A}$ have the following properties: if λ is an eigenvalue then $1/\lambda$ is also an eigenvalue, and if λ is an eigenvalue the complex conjugate λ^* is also an eigenvalue. These properties show that the eigenvalues $\lambda = 1$ and $\lambda = -1$ are always double eigenvalues and that complex eigenvalues with modulus not equal to 1 always appear in quartets.

When all the eigenvalues are on the unit circle the corresponding periodic orbit is stable. If there exist eigenvalues not on the unit circle the periodic orbit is unstable. The different types of instabilities will be studied in detail in the next section.

An interesting question is the following: assume that for $\epsilon = 0$ we know the eigenvalues of the corresponding monodromy matrix $\boldsymbol{A}_0$. What are the possibilities for the eigenvalues of the perturbed matrix $\boldsymbol{A}$, which corresponds to the perturbed system (1) under the symplectic constraint? This question has been answered in the 1950s by Krein [33], Gelfand and Lidskii [34] and Moser [35]. The theory they developed is presented in detail in [32]. According to this theory the only possible movement of simple eigenvalues on the unit circle, due to perturbation, is movement on the unit circle, which means that the stability type of the periodic orbit does not change. On the other hand, multiple eigenvalues on the unit circle have two possibilities. Either they move on the unit circle or off the unit circle. The second case is called complex instability. Finally, if a double eigenvalue equals $+1$ or -1 then under the effect of perturbation these eigenvalues either remain on the unit circle or move on the real axis off the unit circle. The latter case is called in general instability.

An important quantity in determining the fate of eigenvalues under perturbation is the so-called "kind" of the eigenvalues. Let $\boldsymbol{g}$ be an eigenvector corresponding to the simple eigenvalue λ on the unit circle, then we define the inner product

$$\langle \boldsymbol{g}, \boldsymbol{g} \rangle = \mathrm{i}(\boldsymbol{J}\boldsymbol{g}\,\boldsymbol{g}), \tag{10}$$

where $(\cdot, \cdot)$ denotes the usual inner product. Then λ is called an eigenvalue of the first kind if $\langle \boldsymbol{g}, \boldsymbol{g} \rangle > 0$ and of the second kind if $\langle \boldsymbol{g}, \boldsymbol{g} \rangle < 0$. Let now λ be an r-tuple eigenvalue on the unit circle and $\boldsymbol{g}$ a corresponding eigenvector. If $\langle \boldsymbol{g}, \boldsymbol{g} \rangle > 0$ for any eigenvector then λ is called an r-tuple eigenvalue of the first kind and if $\langle \boldsymbol{g}, \boldsymbol{g} \rangle < 0$ then λ is an r-tuple eigenvalue of the second kind. Eigenvalues of the first and second kinds are said to be definite. If on the other hand, there exists an eigenvector $\boldsymbol{g} \neq \boldsymbol{0}$ such that $\langle \boldsymbol{g}, \boldsymbol{g} \rangle = 0$ then the eigenvalue is said to be indefinite or of mixed kind. In this case there exist an eigenvector $\boldsymbol{g}$ so that $\langle \boldsymbol{g}, \boldsymbol{g} \rangle$ is positive and another one so that $\langle \boldsymbol{g}, \boldsymbol{g} \rangle$ is negative. The well-known theorem of Krein–Gelfand–Lidskii [32] states that a linear system is strongly stable (i.e. no small perturbation may turn it unstable) if and only if all eigenvalues lie on the unit circle and are definite.

From the above theory we see that if we have only simple eigenvalues on the unit circle then it is impossible to have instability due to small perturbation. The only way to have instability by perturbing the system is to have two simple eigenvalues of different kinds colliding to create a double eigenvalue. Then instability may occur.

A few remarks on the connection of Hamiltonian systems with symplectic maps are necessary. Since autonomous Hamiltonian systems are conservative, the constancy of the Hamiltonian function (1) introduces a constraint of the form

$$H(q_1, q_2, \ldots, q_{N+1}, p_1, p_2, \ldots, p_{N+1}) = c, \tag{11}$$

where c is a constant value. This constraint fixes an eigenvalue to be equal to 1 and so by the symplectic nature of the problem there must be a second eigenvalue equal to 1. Thus, there are only $2N$ non-constant eigenvalues. So we can constrain the study of a $N + 1$ degrees of freedom Hamiltonian systems to a $2N$-dimensional subspace of the general phase space. This subspace is obtained by the well-known method of the Poincaré surface of section (PSS). Generally speaking we can assume a PSS of the form $q_{N+1} = \text{constant}$. Then only the variables $q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N$ are needed to describe the evolution of an orbit on the PSS, since p_{N+1} can be found by solving Eq. (11). The corresponding monodromy matrix of the periodic orbit is also symplectic and will be denoted as $\boldsymbol{L}$. In this sense a $N+1$ degrees of freedom Hamiltonian system corresponds to $2N$-dimensional symplectic map.

The eigenvalues of $\boldsymbol{L}$ define the stability of the corresponding periodic orbit. These eigenvalues are roots of the characteristic polynomial

$$P(\lambda) = \det(\boldsymbol{L} - \lambda \boldsymbol{I}_{2N}), \tag{12}$$

which is a palindrome of the form [10]

$$P(\lambda) = \lambda^{2N} - A_{N-1}\lambda^{2N-1} + A_{N-2}\lambda^{2N-2} + \cdots + (-1)^N A_0 \lambda^N + \cdots - A_{N-1}\lambda + 1. \tag{13}$$

The coefficients of P can be easily expressed as functions of the elements of matrix $\boldsymbol{L}$. The characteristic polynomial (13) can be written in a simpler form in terms of the stability index

$$b = \frac{1}{\lambda} + \lambda. \tag{14}$$

In particular it becomes

$$Q(b) = A'_0 b^N - A'_1 b^{N-1} + \cdots + (-1)^{N-1} A'_{N-1} b + (-1)^N A'_N. \tag{15}$$

The polynomial $Q(b)$ is called the reduced characteristic polynomial. One of the main advantages of introducing the stability indices b_i, $i = 1, 2, \ldots, N$ is that they solve a polynomial of half the original order, i.e. a polynomial equation of order N. This turns the computational problem into a much more tractable one.

The coefficients A'_i, $i = 0, 1, 2, \ldots, N$ of $Q(b)$ are related to the roots b_i, $i = 1, 2, \ldots, N$ by the well-known formulae

$$\begin{aligned} A'_0 &= 1, \\ A'_1 &= \textstyle\sum_{i=1}^{N} b_i, \\ A'_2 &= \textstyle\sum_{i<j} b_i b_j, \\ &\vdots \\ A'_N &= b_1 b_2, \ldots, b_N. \end{aligned} \tag{16}$$

So A'_i is the sum of all possible i-tuples of $b_1, \ldots, b_n$. The connection between the coefficients A_j, $j = 0, 1, \ldots, N-1$ of the characteristic polynomial (13) and the coefficients A'_i, $i = 0, 1, 2, \ldots, N$ of the reduced characteristic polynomial (15) can be found using some algebra and induction. In particular we get

$$A_{N-i} = \sum_{u=0}^{[i/2]} \binom{N-i+2u}{u} A'_{i-2u}, \quad i = 1, 2, \ldots, N, \tag{17}$$

where $[i]$ denotes the integer part of i and

$$\binom{i}{j}$$

denotes the combinations of i over j. The stability type of a periodic orbit is represented by a point in the N-dimensional parameter space $\mathcal{S}$ whose coordinates are the coefficients $A_0, A_1, \ldots, A_{N-1}$ of the characteristic polynomial $P(\lambda)$.

3. Stability types of Hamiltonian systems with *N+1* degrees of freedom

As explained in the previous section the stability of a periodic orbit in a Hamiltonian system with $N + 1$ degrees of freedom can be studied in the $2N$-dimensional reduced phase space using the method of the Poincaré surface of section. In this sense the Hamiltonian system corresponds to a $2N$-dimensional symplectic map. In the present section we study in detail all the possible stability types that can occur in such a system and the possible transitions between these types.

3.1. Terminology of the possible stability types

The stability type of a periodic orbit is determined by the values of the eigenvalues of the characteristic polynomial (13). Equivalently the stability indices b_i, $i = 1, 2, \dots, N$ (Eq. (14)) can be used to indicate the stability of the orbit in the following way:

(i) The case $b \in (-2, 2)$ corresponds to stability with λ and $1/\lambda$ complex conjugate numbers on the unit circle. In this case we say that the orbit is stable (S).
(ii) The case $b \in (-\infty, -2) \cup (2, \infty)$ corresponds to instability with λ real. In this case we say that the orbit is unstable (U). In particular for $b > 2$ we have two real positive eigenvalues, one (e.g. λ) greater than 1 and the other $1/\lambda$ smaller than 1. For $b < -2$ we have $\lambda < -1$ and $-1 < 1/\lambda < 0$. We remark that these two cases are equivalent regarding the stability character of the periodic orbit, but not completely identical since a positive b cannot become negative under a continuous change of a parameter of the system.
(iii) The case $b \in \mathbb{C} - \mathbb{R}$ corresponds to complex instability (Δ). In this case we have four non-real eigenvalues not laying on the unit circle, forming two pairs of inverse numbers and two pairs of complex conjugate numbers corresponding to two complex conjugate stability indices. Two of the eigenvalues are inside the unit circle while the other two are outside it.

All the above cases are shown in Fig. 1.

In order to describe all the possible stability types of a $(N + 1)$-dimensional Hamiltonian system, we introduce now the following terminology:

Definition 1. We say that the orbit has an "n-tuple stability" if $2n$ eigenvalues are on the unit circle. This stability type is denoted as S_n.

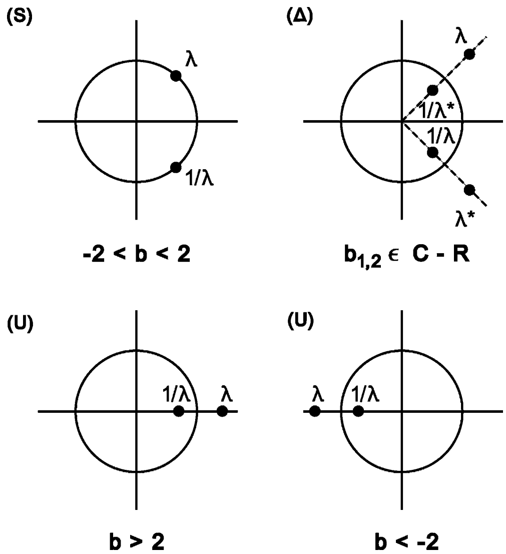

Fig. 1. Configuration of the eigenvalues on the complex plane, with respect to the unit circle, for the stable (S), unstable (U) and complex unstable (Δ) cases. In every case b is the corresponding stability index. We remark that λ^* denotes the complex conjugate of λ.

Definition 2. We say that the orbit has an "m-tuple instability" if $2m$ eigenvalues are on the real axis. This stability type is denoted as U_m.

Definition 3. We say that the orbit has an "l-tuple complex instability" if $4l$ eigenvalues are on the complex plane but not on the unit circle and the real axis. This stability type is denoted as Δ_l.

As mentioned above all the U_m types are not identical due to the different arrangements of the signs of the stability indices. In most applications, when the stability type of a periodic orbit is needed, researchers do not take into account the arrangement of the stability indices on the real axis in the case of instability, but in many cases this is necessary, like for example in finding the possible transitions between different stability types. For a detailed study of the different stability types the following definition is needed.

Definition 4. We say that the orbit has an "(m_1, m_2)-instability" if its stability type is U_m with $m = m_1 + m_2$ and $2m_1$ eigenvalues are negative and $2m_2$ positive. This stability type is denoted as U_{m_1,m_2}.

In the general case, a periodic orbit of a Hamiltonian system with $N+1$ degrees of freedom, or of a $2N$-dimensional symplectic map, has the stability type $S_n U_m \Delta_l$ (or $S_n U_{m_1,m_2} \Delta_l$ if we want to be more specific) where the integers n, $m = m_1 + m_2$ and l satisfy the inequalities

$$0 \le n \le N, \qquad 0 \le m \le N, \qquad 0 \le l \le [N/2], \tag{18}$$

with the constraint

$$n + m + 2l = N. \tag{19}$$

We note that a periodic orbit is stable only when its stability type is S_N. In all other cases the orbit is unstable since there exist eigenvalues not on the unit circle.

3.2. *Counting the possible stability types of Hamiltonian systems with even and odd number of degrees of freedom*

In order to count all the different stability types $S_n U_m \Delta_l$ that a Hamiltonian system with $N+1$ degrees of freedom can exhibit, one must count all the possible combinations of n, m and l. From Eq. (18) we see that for a given N, l can take the values $0, 1, \dots, [N/2]$. For a given value of l, m can take the values $0, 1, \dots, N - 2l$. Then the value of n is determined by Eq. (19). So the number $\mathcal{N}$ of all the possible stability types $S_n U_m \Delta_l$ of a Hamiltonian system with $N + 1$ degrees of freedom is

$$\mathcal{N} = \sum_{l=0}^{[N/2]} \sum_{m=0}^{N-2l} 1 = \begin{cases} \frac{1}{4}(N+1)(N+3), & \text{for } N+1 \text{ even}, \\ \frac{1}{4}(N+2)^2, & \text{for } N+1 \text{ odd}, \end{cases} \tag{20}$$

since the value of $[N/2]$ is $(N-1)/2$ for $N+1$ even and $N/2$ for $N+1$ odd.

The different stability types correspond to different regions of the N-dimensional parameter space $\mathcal{S}$ whose coordinates are the coefficients $A_0, A_1, \dots, A_{N-1}$ of the characteristic polynomial $P(\lambda)$. These coefficients can be expressed as functions of the elements of the monodromy matrix $\boldsymbol{L}$ and through Eq. (17) as functions of the coefficients $A'_0, A'_1, \dots, A'_N$ of the reduced characteristic polynomial $Q(b)$. The stability type U_m corresponds to $m + 1$ different regions of the parameter space since there exist $m + 1$ different arrangements of the signs of the stability indices b_i on the real axis. In particular U_m corresponds to the cases $U_{0,m}, U_{1,m-1}, U_{2,m-2}, \dots, U_{m,0}$. So the number $\mathcal{N}^*$ of different stability regions in the N-dimensional parameter space $\mathcal{S}$ is

$$\mathcal{N}^* = \sum_{l=0}^{[N/2]} \sum_{m=0}^{N-2l} (m+1) = \begin{cases} \frac{1}{24}(N+1)(N+3)(2N+7), & \text{for } N+1 \text{ even}, \\ \frac{1}{24}(N+2)(N+4)(2N+3), & \text{for } N+1 \text{ odd}. \end{cases} \tag{21}$$

A significant difference between $(N+1)$-dimensional Hamiltonian systems with odd or even degrees of freedom is that in the former case the system can be completely complex unstable ($\Delta_{N/2}$), while in the latter this is not possible since the maximum value of l is $(N-1)/2$.

3.3. Transitions between different stability types

In Fig. 2 the basic transitions between different stability types are shown schematically. These transitions are as follows:

(i) The transition $S_1 \to U_1$. This transition happens when two eigenvalues move on the unit circle, as a parameter of the Hamiltonian system changes, coincide on $\lambda = 1$ and split along the positive real axis (tangent bifurcation) (Fig. 2(a)). The corresponding stability index is positive and increases through $b = 2$. The final stability type is $U_{0,1}$. A similar transition from stability to instability, happens when the two eigenvalues leave the unit circle passing though the point $\lambda = -1$ (period-doubling bifurcation). The stability index decreases through $b = -2$ and the final state is $U_{1,0}$.

(ii) The transition $S_2 \to \Delta_1$. This transition happens when two pairs of eigenvalues moving on the unit circle collide and split off it, at a point where $\lambda^2 \neq 1$ (Fig. 2(b)). The corresponding real stability indices $-2 < b_1 < 2$, $-2 < b_2 < 2$ become equal $b_1 = b_2$ and then become complex.

(iii) The transition $U_2 \to \Delta_1$. This transition happens when two real positive (or negative) pairs of eigenvalues become equal and move on the complex plane, leaving the real axis and staying away from the unit circle (Fig. 2(c)). The corresponding real stability indices $b_1 > 2$, $b_2 > 2$ (or $b_1 < -2$, $b_2 < -2$) become equal $b_1 = b_2$ and then become complex. So we have the transition $U_{0,2} \to \Delta_1$ (or $U_{2,0} \to \Delta_1$). It is evident from the configuration of the eigenvalues on the plane, that the transition $U_{1,1} \to \Delta_1$ is not possible, since the two stability indices cannot become equal.

(iv) The transition $S_1U_1 \to \Delta_1$. This transition happens when two pairs of eigenvalues coincide on $\lambda = 1$ (or $\lambda = -1$) and split on the complex plane, staying away from the unit circle. The pair corresponding to the S_1 case was initially on the unit circle, while, the one corresponding to the U_1 case was on the real positive (or negative) axis (Fig. 2(d)). The stability indices that were real initially $-2 < b_1 < 2$, $b_2 > 2$ (or $-2 < b_1 < 2$, $b_2 < -2$) became equal $b_1 = b_2 = 2$ (or -2) and then complex. This transition is of the form $S_1U_{0,1} \to \Delta_1$ (or $S_1U_{1,0} \to \Delta_1$).

We remark that the above transitions can also happen following the opposite direction $U_1 \to S_1$, $\Delta_1 \to S_2$, $\Delta_1 \to U_2$ and $\Delta_1 \to S_1U_1$.

As already mentioned the stability type of a periodic orbit of a Hamiltonian system is represented by a point A in the corresponding N-dimensional parameter space $\mathcal{S}$. The coordinates of A are the coefficients of the characteristic polynomial $P(\lambda)$ so, $A \equiv (A_0, A_1, \ldots, A_{N-1})$. As a parameter of the Hamiltonian system changes the coefficients of the characteristic polynomial $P(\lambda)$ also change, causing possible changes in the stability type of the periodic orbit and the motion of point A in $\mathcal{S}$. As a result of that the stability indices b_i, $i = 1, 2, \ldots, N$ change too.

The above described transitions happen when certain constraints on the values of the stability indices, are valid. These constraints define a transition hypersurface in the parameter space $\mathcal{S}$, the crossing of which, by point A, corresponds to the change of the stability type of the orbit. The equation of this hypersurface is obtained by

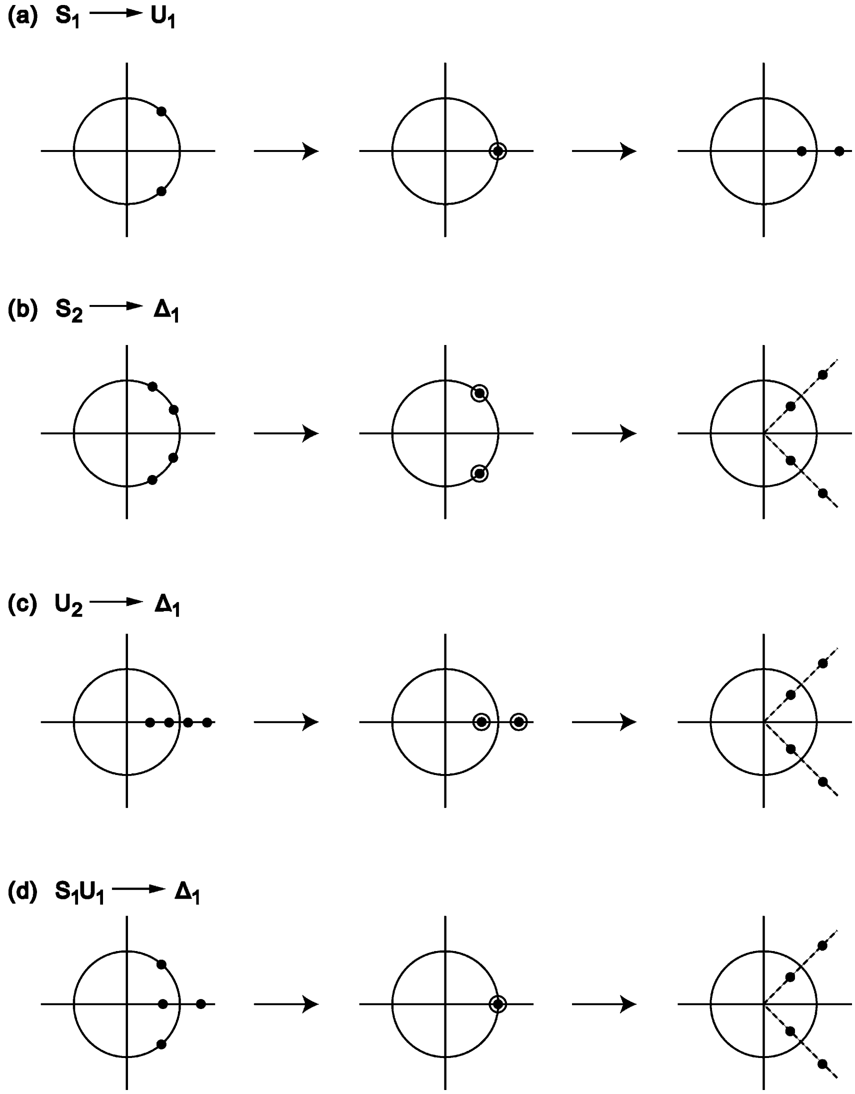

Fig. 2. Schematic representations of the configuration of the eigenvalues on the complex plane for the basic transitions between different stability types, where one or two pairs of eigenvalues are involved. In every panel the unit circle is also plotted. (a) $S_1 \to U_1$. Two eigenvalues move on the unit circle, coincide on $\lambda = 1$ and split along the positive real axis. (b) $S_2 \to \Delta_1$. Two pairs of eigenvalues move on the unit circle, collide and split off it at a point where $\lambda^2 \neq 1$. (c) $U_2 \to \Delta_1$. Two real pairs of eigenvalues coincide and move on the complex plane, but not on the unit circle. (d) $S_1U_1 \to \Delta_1$. A pair of eigenvalues on the unit circle collide with a pair of real eigenvalues on $\lambda = 1$ and split on the complex plane staying away from the unit circle and the real axis.

substituting the constraints on the stability indices b_i in Eqs. (16) and (17). For instance the transition $S_1 \to U_1$ (Fig. 2(a)) happens when b passes through $b = 2$ for $S_1 \to U_{0,1}$ or $b = -2$ for $S_1 \to U_{1,0}$, which corresponds to A crossing the $(N-1)$-dimensional hypersurface in $\mathcal{S}$ produced by putting $b = 2$ or $b = -2$ in Eqs. (16) and (17). In a similar way $S_2 \to \Delta_1$ (Fig. 2(b)) and $U_2 \to \Delta_1$ (Fig. 2(c)) happen when point A crosses the $(N-1)$-dimensional hypersurface produced by $b_1 = b_2$, while the transition $S_1U_1 \to \Delta_1$ (Fig. 2(d)) happens when A crosses the $(N-2)$-dimensional hypersurface produced by $b_1 = b_2 = 2$ or $b_1 = b_2 = -2$.

We remark that all the possible constraints are of the form $b = +2$, $b = -2$, $b_1 = b_2$, $b_1 = b_2 = +2$ and $b_1 = b_2 = -2$. These constraints define through Eqs. (16) and (17) the dimensionality of the corresponding hypersurface. Every constraint reduces the dimensionality of the hypersurface by constant number. In particular, the constraints $b = +2$, $b = -2$, $b_1 = b_2$ decrease the dimensionality of the hypersurface they define by 1, while the constraints $b_1 = b_2 = +2$, $b_1 = b_2 = -2$ by 2.

3.3.1. Direct transitions of the form $S_n U_m \Delta_l \to S_{n+\delta n} U_{m+\delta m} \Delta_{l+\delta l}$

We consider now the problem of finding all the possible direct transitions between different stability types and the dimension of the corresponding transition hypersurface in $\mathcal{S}$. The different possible arrangements of the eigenvalues of the U_m case makes this problem complicated, since certain arrangements of the eigenvalues can even make a transition impossible. As an example we remind the already mentioned case of $U_2 \to \Delta_1$. In this case $U_{2,0} \to \Delta_1$ and $U_{0,2} \to \Delta_1$ can happen when point A crosses the $(N-1)$-dimensional hypersurface produced by the constraint $b_1 = b_2$, while the transition $U_{1,1} \to \Delta_1$ is impossible. As a first step we find if and how a transition of the form

$$S_n U_m \Delta_l \to S_{n+\delta n} U_{m+\delta m} \Delta_{l+\delta l}, \tag{22}$$

where δn, δm, δl, are the changes in the multiplicity of S, U and Δ, can happen. By this we mean that there exist at least one configuration of the eigenvalues, compatible with the U_m and $U_{m+\delta m}$ types, which allows the transition to happen.

Since both the initial ($S_n U_m \Delta_l$) and the final ($S_{n+\delta n} U_{m+\delta m} \Delta_{l+\delta l}$) stability types satisfy the constraint (19) we conclude that δn, δm, δl satisfy

$$\delta n + \delta m + 2\delta l = 0. \tag{23}$$

So the constraints (18), (19) and (23), define all the possible direct transitions of the form (22) of a Hamiltonian system with $N + 1$ degrees of freedom.

The different arrangements of the eigenvalues, influence the exact form of the constraints that define the hypersurface in $\mathcal{S}$, which corresponds to the transition (22). For example, the transition $S_1 \leftrightarrow U_1$ introduce the constraint $b = 2$ or $b = -2$, which decreases the dimensionality of the hypersurface by 1, with respect to the dimension of $\mathcal{S}$. In order to simplify the study of the transitions we consider equivalent the constraints $b = +2$, $b = -2$, in the sense that they are of the same form and have the same effect on the dimensionality of the hypersurface that represents the transition in the parameter space $\mathcal{S}$.

Based on the simple transitions shown in Fig. 2 we find the constraints on the stability indices b_i that define the corresponding transition hypersurface in the parameter space $\mathcal{S}$. In particular we have the two cases seen in Table 1:

(i) $\delta n \cdot \delta m \leq 0$. In this case the multiplicity of S or U does not change or if both of them change the changes have different signs. The change δl is found from Eq. (23). We distinguish the following cases:

Table 1
The number of different kinds of constraints and the dimension D of the corresponding transition hypersurface in the N-dimensional parameter space $\mathcal{S}$, for the possible transitions $S_n U_m \Delta_l \to S_{n+\delta n} U_{m+\delta m} \Delta_{l+\delta l}$

Case	Number of constraints			D
	$b_1 = b_2$	$b_1 = \pm 2$	$b_1 = b_2 = \pm 2$	
(i) $\delta n \cdot \delta m \leq 0$	$\|\delta l\|$	$\|\delta l + \delta n\| - \|\delta l\|$	0	$N - \|\delta l + \delta n\|$
(ii(a)) $\delta n \cdot \delta m > 0$ with δn, δm even	$\|\delta l\|$	0	0	$N - \|\delta l\|$
(ii(b)) $\delta n \cdot \delta m > 0$ with δn, δm odd	$\|\delta l\| - 1$	0	1	$N - \|\delta l\| - 1$

(a) If $\delta m = 0$ Eq. (23) gives $\delta n = -2\delta l$, which means that $|\delta l|$ quartets of eigenvalues move from the unit circle to the complex plane, but not on the real axis (or vice versa), following $|\delta l|$ transitions of the form seen in Fig. 2(b). These transitions introduce $|\delta l|$ constraints of the form $b_1 = b_2$.

(b) If $\delta n = 0$ we get $\delta m = -2\delta l$, which means that $|\delta l|$ quartets of eigenvalues move from the real axis to the complex plane, but not on the unit circle (or vice versa), following $|\delta l|$ transitions of the form seen in Fig. 2(c). These transitions introduce $|\delta l|$ constraints of the form $b_1 = b_2$.

(c) If $\delta n \cdot \delta m < 0$ and $\delta l = 0$ we get from Eq. (23) $\delta n = -\delta m$, which means that $|\delta n|$ pairs of eigenvalues move from the unit circle to the real axis (or vice versa), following $|\delta n|$ transitions of the form seen in Fig. 2(a). These transitions introduce $|\delta n|$ constraints of the form $b = 2$ or $b = -2$.

(d) If $\delta n \cdot \delta m < 0$ and $\delta l \neq 0$ then δl has the same sign with δn or δm. If $\delta n \cdot \delta l > 0$ the eigenvalues come from (or go to) the real axis, following $|\delta n|$ transitions of the form of Fig. 2(a) and $|\delta l|$ transitions of the form of Fig. 2(c). These transitions introduce $|\delta n|$ constraints $b = \pm 2$ and $|\delta l|$ constraints $b_1 = b_2$, respectively. If $\delta m \cdot \delta l > 0$ the eigenvalues come from (or go to) the unit circle, following $|\delta m|$ transitions of the form of Fig. 2(a) and $|\delta l|$ transitions of the form of Fig. 2(b). These transitions introduce $|\delta m|$ constraints $b = \pm 2$ and $|\delta l|$ constraints $b_1 = b_2$, respectively.

One can easily check that all the above transitions introduce $|\delta l + \delta n| - |\delta l|$ constraints of the form $b = \pm 2$ (note that always $|\delta l + \delta n| = |\delta l + \delta m|$) and $|\delta l|$ constraints of the form $b_1 = b_2$. All these constraints are independent to each other since they refer to different stability indices. So the dimension D of the corresponding transition hypersurface is

$$D = N - |\delta l + \delta n|. \tag{24}$$

(ii) $\delta n \cdot \delta m > 0$. In this case the multiplicity of both S and U increase (or decrease) leading to decrement (or increment) of the multiplicity of Δ. This means that eigenvalues come from (or go to) the complex plane in quartets. From Eq. (23) we have $\delta n + \delta m = -2\delta l$. We distinguish the following cases:

(a) If δn and δm are even, the transition happens as seen in cases (b) and (c) of Fig. 2. So $|\delta l|$ constraints of the form $b_1 = b_2$ are introduced and the dimension D of the corresponding transition hypersurface is

$$D = N - |\delta l|. \tag{25}$$

(b) If δn and δm are odd then one transition of the form shown in Fig. 2(d) is needed. Thus, we get $|\delta l| - 1$ constraints of the form $b_1 = b_2$ and 1 constraint of the form $b_1 = b_2 = \pm 2$. The dimension D of the transition hypersurface is

$$D = N - |\delta l| - 1. \tag{26}$$

From the above analysis it is evident that the direct transition between any two stability types permissible by Eqs. (18) and (19) is possible. The dimension D of the transition hypersurface given by Eqs. (24)–(26) is the maximal possible in the sense that any particular arrangement of the eigenvalues of U_m and $U_{m+\delta m}$ which is compatible with the transition (22) is performed (if it happens at all) by the crossing of a M-dimensional hypersurface with

$$M \leq D. \tag{27}$$

The dimension D of the hypersurface which corresponds to a certain transition is an indicator of how probable this transition is, or in other words how specific the parameters that influence the stability of an orbit must be in order for this transition to happen. For example, the transition $S_N \to U_N$, that destabilizes a fully stable orbit making it fully unstable, is done when the point A (which represents the stability state of the orbit in the parameter space $\mathcal{S}$) passes through a specific point in $\mathcal{S}$. In this case we have $\delta n = -N$ and $\delta m = N$ and from Eq. (24) we get

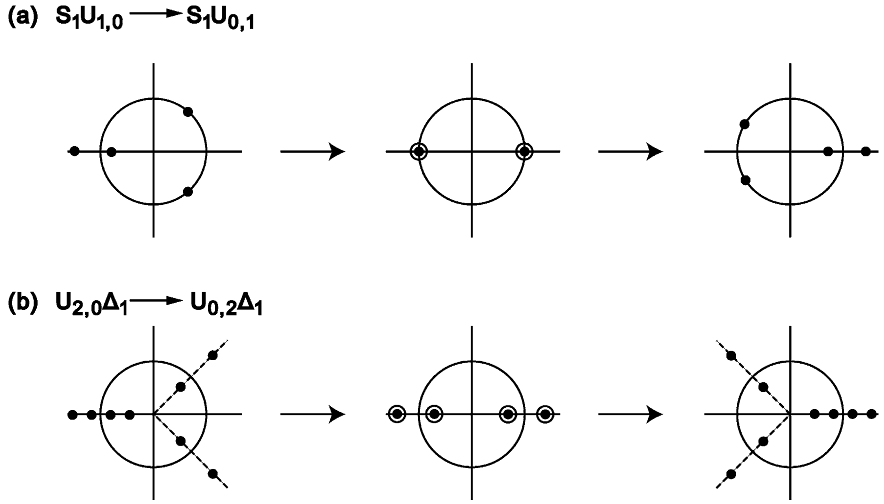

Fig. 3. Simple transitions between different stability types where one or two real and negative pairs of eigenvalues become positive, changing the distribution of the eigenvalues in the U_m stability type without changing m. (a) $S_1U_{1,0} \to S_1U_{0,1}$. Two real negative eigenvalues move on the unit circle passing trough $\lambda = -1$ while simultaneously two eigenvalues from the unit circle pass from $\lambda = 1$ and split along the positive real axis. (b) $U_{2,0}\Delta_1 \to U_{0,2}\Delta_1$. Two pairs of negative real eigenvalues coincide on the real axis and split on the complex plane, staying away from the unit circle, while simultaneously four complex eigenvalues, not laying on the unit circle, fall on the real positive axis and split remaining on it.

$D = 0$. So, this transition is performed when the parameters of the dynamical system have very specific values, in order for point A to pass through a particular point in the N-dimensional space $\mathcal{S}$. On the other hand, the transition $S_N \to S_{N-1}U_1$ is more easily performed, since it happens when the point A crosses a transition hypersurface of dimension $N-1$ as it can be easily seen from Eq. (24).

3.3.2. Direct transitions of the form $S_nU_{m_1,m_2}\Delta_l \to S_{n'}U_{m'_1,m'_2}\Delta_{l'}$

In order to study the possible transitions between different stability types, taking into account the different arrangements of the eigenvalues in U_m, we must consider transitions that change the arrangement of the eigenvalues in U_m in addition to the ones described in Fig. 2. Simple transitions that change the internal distribution of the eigenvalues in the U_m stability type are seen in Fig. 3. In particular in Fig. 3(a) we see the transition $S_1U_{1,0} \to S_1U_{0,1}$, which happens when two real negative eigenvalues move on the unit circle passing trough $\lambda = -1$ and simultaneously two eigenvalues from the unit circle pass from $\lambda = 1$ and split along the positive real axis. The stability indices become simultaneously $b_1 = -2$ and $b_2 = 2$. So the transfer of a pair of real eigenvalues from negative to positive values (and vice versa) is done through another pair initially located on the unit circle. In order to transfer two pairs of real eigenvalues from negative to positive values (and vice versa) we perform the above transition twice, which means that two pairs of eigenvalues on the unit circle are needed. In this case we have $S_2U_{2,0} \to S_2U_{0,2}$ and the introduced constraints are $b_1 = b_2 = -2$ and $b_3 = b_4 = 2$. Another way to perform the same transfer of eigenvalues is through a quartet of eigenvalues corresponding to case Δ_1. In particular we can have $U_{2,0}\Delta_1 \to U_{0,2}\Delta_1$ as seen in Fig. 3(b). In this case two pairs of negative real eigenvalues coincide on the real axis and then move on the complex plane but not on the unit circle, while simultaneously the four eigenvalues that correspond to Δ_1, fall on the real positive axis and split remaining on it. So we have the constrains $b_1 = b_2 < 0$ and $b_3 = b_4 > 0$ which are more general than the constrains $b_1 = b_2 = -2$ and $b_3 = b_4 = 2$ we get for the transition $S_2U_{2,0} \to S_2U_{0,2}$. This means that the transition hypersurface which corresponds to the transition $S_2U_{2,0} \to S_2U_{0,2}$ is a subset of the hypersurface which corresponds to $U_{2,0}\Delta_1 \to U_{0,2}\Delta_1$.

Table 2
The 10 possible simple transitions of the stability type $S_n U_{m_1,m_2} \Delta_l \equiv [n, m_1, m_2, l]$ to the final stability types listed, when only one or two pairs of eigenvalues are transferred. Any other general transition can be decomposed in successive applications of the transitions shown in this table, taking care of not using the same eigenvalues twice. The constraints on the values of the stability indices that any particular transition introduces are registered, as well as the decrement d of the dimensionality of the corresponding transition hypersurface

Case	Final type	Constraints	d
1	$[n, m_1 \mp 2, m_2, l \pm 1]$	$b_1 = b_2$	1
2	$[n, m_1, m_2 \mp 2, l \pm 1]$	$b_1 = b_2$	1
3	$[n \mp 1, m_1 \pm 1, m_2, l]$	$b_1 = -2$	1
4	$[n \mp 1, m_1, m_2 \pm 1, l]$	$b_1 = 2$	1
5	$[n \mp 1, m_1 \mp 1, m_2, l \pm 1]$	$b_1 = b_2 = -2$	2
6	$[n \mp 1, m_1, m_2 \mp 1, l \pm 1]$	$b_1 = b_2 = 2$	2
7	$[n, m_1 \mp 1, m_2 \mp 1, l \pm 1]$	$b_1 = 2$, $b_2 = -2$ and $b_3 = 2$ or -2, $n \geq 1$	3
8	$[n, m_1 \mp 2, m_2 \pm 2, l]$	$b_1 = b_2$, $b_3 = b_4$, $l \geq 1$	2
9	$[n, m_1 \mp 1, m_2 \pm 1, l]$	$b_1 = -2$, $b_2 = 2$, $n \geq 1$	2
10	$[n \mp 2, m_1, m_2, l \pm 1]$	$b_1 = b_2$	1

Based on the cases described in Figs. 2 and 3, and using for simplicity the notation

$$S_n U_{m_1,m_2} \Delta_l \equiv [n, m_1, m_2, l], \tag{28}$$

we register in Table 2, all the possible transitions which involve one or two pairs of eigenvalues. In cases 1 and 2 four negative or positive real eigenvalues move from the real axis to the complex plain, but not on the unit circle (or vice versa) following transitions of the form seen in Fig. 2(c). These transitions introduce a $b_1 = b_2$ constraint which decreases the dimensionality of the corresponding transition hypersurface by $d = 1$. In cases 3 and 4 a pair of negative or positive eigenvalues move from the real axis to the unit circle (or vice versa) following transitions of the form seen in Fig. 2(a). These transitions introduce a $b = -2$ or $b = 2$ constraint, respectively, with $d = 1$. In cases 5 and 6 a pair of negative or positive real eigenvalues and a pair of eigenvalues located initially on the unit circle move to the complex plain, staying away from the unit circle and the real axis (or vice versa), following transitions of the form seen in Fig. 2(d). These transitions introduce a $b_1 = b_2 = -2$ or $b_1 = b_2 = 2$ constraint, respectively, with $d = 2$. In case 7 a pair of negative and a pair of positive real eigenvalues move on the complex plain staying away from the unit circle and the real axis (or vice versa). In order for this transition to happen a pair of eigenvalues on the unit circle is needed. This pair goes to the real negative (or positive) axis, following a transition similar to the one of Fig. 2(a), introducing the constraint $b = -2$ (or $b = 2$). Simultaneously four eigenvalues come from the complex plain, coincide at point $\lambda = 1$ (or $\lambda = -1$) and one pair split on the real positive (or negative) axis while the other move on the unit circle (opposite transition to the one seen in Fig. 2(d)). This transition introduce the constraint $b_1 = b_2 = 2$ (or $b_1 = b_2 = -2$). So in case 7 of Table 2 we have $d = 3$. In case 8 two pairs and in case 9 one pair of real eigenvalues change their position in U_m, following the transitions of Fig. 3(b) and (a), respectively. In case 8 a quartet of eigenvalues on the complex plain and in case 9 a pair of eigenvalues on the real axis are needed. In both cases we have $d = 2$. In case 10 two pairs of eigenvalues move from the unit circle to the complex plain (or vice versa) following the transition of Fig. 2(b). In this case a constraint $b_1 = b_2$ is introduced, so $d = 1$.

When a transition between different stability types occurs pairs and/or quartets of eigenvalues are simultaneously transferred between the S, U and Δ types. The main difficulties in studying these transitions are the different arrangements of the eigenvalues in U_m and the fact that eigenvalues come from (or go to) Δ_l in quartets. The transitions in Table 2 are ordered by taking into account these difficulties. The first cases listed are simple cases with $d = 1$. So m_1 or m_2 change in cases 1 and 2 by changing l and in cases 3 and 4 by changing n. In cases 5–7 m_1

or/and m_2 change by moving a quartet of eigenvalues from Δ_l, in a more complicated way since $d > 1$. In cases 8 and 9 m_1 and m_2 change while n and l remain unchanged. In the final case 10 m_1 and m_2 do not change, while l and n do.

In order to determine if a transition from an initial stability type $[n, m_1, m_2, l]$ to a final type $[n', m'_1, m'_2, l']$ is possible, and also find the dimension of the corresponding transition hypersurface in $\mathcal{S}$, we decompose the general transition in several simple transitions from the ones registered in Table 2, taking care of not using the same pair or quartet of eigenvalues twice. Every time one of the transitions of Table 2 is used the corresponding constraint reduces the dimensionality of the hypersurface by d. So, the procedure for finding the dimension of the transition hypersurface can be described as follows. We change the initial stability type $[n, m_1, m_2, l]$ applying case 1 of Table 2 as many times as possible. Then we change the resulting stability type applying case 2 of Table 2 as many times as possible and so on. In performing these steps we take into account the following rules:

(i) The quantities $\delta n = n' - n$, $\delta m_1 = m'_1 - m_1$, $\delta m_2 = m'_2 - m_2$, $\delta l = l' - l$ must keep their initial sign or become equal to 0.
(ii) If $\delta n = 0$ the cases of Table 2 that change n cannot be applied. The same holds for δm_1, δm_2 and δl.
(iii) Any step is performed only if it brings the stability type closer to its final form in the sense that the quantity $\Sigma_\delta = |\delta n| + |\delta m_1| + |\delta m_2| + |\delta l|$ decreases.
(iv) Any step is performed only if the pairs or quartets of eigenvalues involved have not been used previously.
(v) The procedure stops if $\Sigma_\delta = 0$, which means that the final stability type has been reached and the general transition is possible. If we reach at the bottom of Table 2, apply case 10 and $\Sigma_\delta \neq 0$ then the general transition is not possible.

The following simple example clarifies the use of Tables 1 and 2. We consider a Hamiltonian system with six degrees of freedom so that $N = 5$ and the general transition $S_1U_2\Delta_1 \rightarrow S_1U_4$. Using Table 1 and the fact that $\delta n = 0$, $\delta m = 2$, $\delta l = -1$, we conclude that there exist at least one arrangement of the eigenvalues that makes this transition possible, through a hypersurface with maximum dimension $D = 4$ (Eq. (24)). For example, the transition

$$[1(1), 0(0), 2(2), 1(1)] \rightarrow [1(1), 0(0), 4(2), 0(0)]$$

can happen following case 2 of Table 2. Case 1 of Table 2 cannot be applied since it is against rule (ii). This transition is performed in the five-dimensional parametric space $\mathcal{S}$, through a four-dimensional hypersurface $\mathcal{F}_1$ produced by the constraint $b_1 = b_2$. We note that in the parentheses beside the indices we register the number of pairs (or quartets for the l index) of eigenvalues that can be moved, in order to easily check the validity of rule (iv).

Another arrangement of the eigenvalues compatible with the general form of the transition is

$$[1, 2, 0, 1] \rightarrow [1, 1, 3, 0].$$

This transition can be performed by applying successively cases 2 and 9 of Table 2, i.e.

$$[1(1), 2(2), 0(0), 1(1)] \rightarrow [1(1), 2(2), 2(0), 0(0)] \rightarrow [1(0), 1(1), 3(0), 0(0)].$$

All the other cases cannot be applied since they are against rule (iii) (cases 1, 8) and rule (ii) (cases 3–7). The corresponding constraints are $b_1 = b_2$, $b_3 = -2$ and $b_4 = 2$ which means that this transition is performed through a two-dimensional surface $\mathcal{F}_2$. It is obvious that $\mathcal{F}_2 \subset \mathcal{F}_1$. On the other hand, the transition

$$[1, 2, 0, 1] \rightarrow [1, 0, 4, 0]$$

cannot happen. For example, we could have

$$[1(1), 2(2), 0(0), 1(1)] \rightarrow [1(1), 2(2), 2(0), 0(0)] \rightarrow [1(0), 1(1), 3(0), 0(0)]$$

following cases 2 and 9 of Table 2 and stop because we cannot apply case 9 again, since the pair of eigenvalues on the unit circle, that corresponds to S_1, has already been used in the previous step (rule (iv)). Also case 10 cannot be applied since $\delta l = 0$ (rule (ii)).

From the above example it becomes clear how to use Tables 1 and 2 in order to find the possible transitions between different stability types by taking or not into account the possible arrangements of the eigenvalues related to U_m. Also the equations that define the transition hypersurface in the N-dimensional parameter space $\mathcal{S}$ are obtained along with its dimensionality.

4. Cases of definite number of degrees of freedom

We apply the results of the previous sections in some particular cases, namely to Hamiltonian systems with two, three and four degrees of freedom. Although the stability properties of Hamiltonian systems with two and three degrees of freedom are well known [8–10] we include a brief treatment of these cases for completeness sake and in order to illustrate the use of the terminology of the different stability types introduced in Section 3.1. In addition, these well-known cases give us a good opportunity to check the information provided in Tables 1 and 2. On the other hand, the study of systems with four degrees of freedom is far from considered complete. So we do a detailed study of all the possible stability types and the transitions between them.

4.1. The case of two degrees of freedom

In the case of a Hamiltonian system with two degrees of freedom we have $N + 1 = 2$ so the characteristic polynomial (13) becomes

$$P(\lambda) = \lambda^2 - A_0\lambda + 1, \tag{29}$$

where the coefficient A_0 is the trace of the monodromy matrix $\boldsymbol{L}$

$$A_0 = \mathrm{Tr}\,\boldsymbol{L}. \tag{30}$$

So, there exists only one stability index b_1, hence the reduced characteristic polynomial (15) is simply

$$Q(b) = b - A_1', \tag{31}$$

where

$$A_1' = b_1 = A_0, \tag{32}$$

as we derive from Eqs. (16) and (17), for $N = 1$. So the stability type of the orbit depends on the value of A_0. In particular we have only two types of stability, in agreement to Eq. (20) for $N = 1$. If $|b_1| = |A_0| < 2$ the orbit is stable (S_1) and if $|A_0| > 2$ unstable (U_1). The usual notation for these two stability types are S for stable and U for unstable [36–39] which does not practically differ from our notation.

The parameter space $\mathcal{S}$, where the possible stability types are defined is one-dimensional since the only coordinate is A_0. In this space we have three different stability regions, in agreement to Eq. (21) for $N = 1$. The transition boundaries correspond to the constraints $b_1 = -2$ and $b_1 = 2$, so that the possible regions are: (a) $A_0 < -2$, which corresponds to the $U_{1,0}$ stability type, (b) $-2 < A_0 < 2$, which corresponds to the S_1 stability type, and (c) $A_0 > 2$ which corresponds to the $U_{0,1}$ stability type. The only possible transition is $S \to U$ ($U_{1,0}$ or $U_{0,1}$) and vice versa, which corresponds to case 1 of Table 1 since $\delta m \cdot \delta n \leq 0$. Since $\delta l = 0$ these transitions are performed when the point A, that represents the stability state of the periodic orbit, passes through the points $A_0 = \pm 2$ in agreement to case 1 of Table 1. The transition $U_{1,0} \to U_{0,1}$ is not possible since the two regions in $\mathcal{S}$ are not neighboring. Also this transition does not correspond to any of the cases listed in Table 2, since case 9 which resembles to the transition $U_{1,0} \to U_{0,1}$ is valid for $n \geq 1$.

4.2. The case of three degrees of freedom

In this case $N + 1 = 3$ so the characteristic polynomial (13) is

$$P(\lambda) = \lambda^4 - A_1\lambda^3 + A_0\lambda^2 - A_1\lambda + 1. \tag{33}$$

The coefficient A_0 and A_1 are related to the elements of the monodromy matrix $\boldsymbol{L}$ through the relations

$$A_0 = \sum_{i<j} \begin{vmatrix} L_{ii} & L_{ij} \\ L_{ji} & L_{jj} \end{vmatrix}, \qquad A_1 = \mathrm{Tr}\,\boldsymbol{L}. \tag{34}$$

The two stability indices b_1, b_2 are roots of the reduced characteristic polynomial (15)

$$Q(b) = b^2 - A'_1 b + A'_2 = 0. \tag{35}$$

The coefficients of the two polynomials are related through

$$A'_1 = b_1 + b_2 = A_1, \qquad A'_2 = b_1 b_2 = A_0 - 2, \tag{36}$$

as we derive from Eqs. (16) and (17) for $N = 2$.

The parameter space $\mathcal{S}$ where the possible stability types are defined, is two-dimensional with coordinates the coefficients A_0 and A_1. The transition boundaries between the different stability regions in $\mathcal{S}$ are given by substituting $b = \pm 2$ in Eq. (35) and using Eq. (36), which yields the lines

$$b = +2 \Rightarrow A_0 = 2A_1 - 2, \qquad b = -2 \Rightarrow A_0 = -2A_1 - 2 \tag{37}$$

and by forcing $b_1 = b_2$ in Eq. (36) or equivalently putting the discriminant of Eq. (35) equal to zero, which yields the curve

$$A_0 = \tfrac{1}{4}A_1^2 + 2. \tag{38}$$

These boundaries create seven regions of different stability types in agreement to Eq. (21) for $N = 2$. These regions are seen in Fig. 4.

We remark that the above analysis was performed by Broucke [9] in his work on the elliptic restricted three-body problem, where he introduced a notation different to ours for the possible stability types. He named even–even instability the type $U_{0,2}$, even–odd instability the type $U_{1,1}$, odd–odd instability the type $U_{2,0}$, even–semi-instability the type $S_1U_{0,1}$ and odd–semi-instability the type $S_1U_{1,0}$, while he named stable the S_2 type and complex unstable the Δ_1 type. We underline the fact that Broucke gave different names for all the regions seen in Fig. 4 taking into account the different configuration of the eigenvalues on the real axis. In a similar approach Dullin and Meiss [40] named the S_1 type elliptic denoting it as "E", the $U_{0,1}$ type hyperbolic denoting it as "H", the $U_{1,0}$ type inverse hyperbolic denoting it as "I" and the Δ_1 type complex denoting it as "CQ". Using the above notation they denoted the seven different stability types of a four-dimensional symplectic map as EE (S_2), EH ($S_1U_{0,1}$), EI ($S_1U_{1,0}$), II ($U_{2,0}$), HI ($U_{1,1}$), HH ($U_{0,2}$) and CQ (Δ_1).

On the other hand, most authors do not take into account the different arrangements of the eigenvalues on the real axis, referring only to four different cases in agreement to Eq. (20) for $N = 2$ [13–30]. Usually the stable

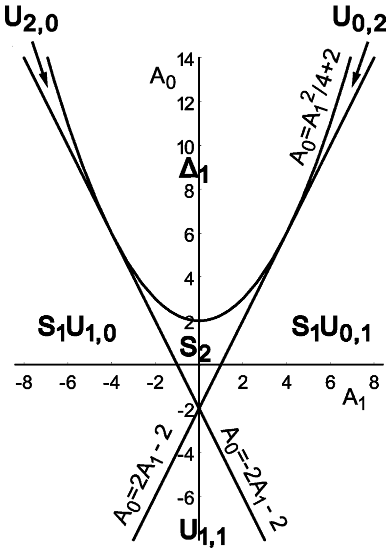

Fig. 4. The parameter space $\mathcal{S}$ of a Hamiltonian system with three degrees of freedom. A_0 and A_1 are the coefficients of the characteristic polynomial. The boundaries of the different stability regions are the lines $A_0 = 2A_1 - 2$, $A_0 = -2A_1 - 2$, which correspond to one stability index being equal to 2 and -2, respectively, and the line $A_0 = (A_1^2/4) + 2$ which corresponds to two stability indices being equal to each other. On every stability region the corresponding stability type is marked.

case is denoted by S and the complex unstable case by Δ, similar to our S_2 and Δ_1 notations. The simple unstable case (U) corresponds to S_1U_1 and in particular to cases $S_1U_{1,0}$ and $S_1U_{0,1}$, while the double unstable case (DU) corresponds to U_2 and in particular to cases $U_{2,0}$, $U_{1,1}$ and $U_{0,2}$ [18,19,21,22,28,29]. Some authors follow the above distinction of stability types using sometimes different terminology. As an example we refer to the papers of Vrahatis et al. [23,24] where the authors name as elliptic–elliptic (EE) the stable case, elliptic–hyperbolic (EH) the case U, hyperbolic–hyperbolic (HH) the DU case and complex unstable (CU) the Δ case.

Although our terminology may seem a little heavy for the cases of two and three degrees of freedom, in comparison to the already used terminology, it is perfectly suited for systems with many degrees of freedom, since it gives in a very clear way all the information needed for the configuration of the eigenvalues on the complex plane.

4.3. Complete study of the four degrees of freedom case

We now continue with the main concept of this section, which is to find the stability regions in the three-dimensional parameter space $\mathcal{S}$, of a four degrees of freedom Hamiltonian system. In this case we have $N = 3$ and the characteristic polynomial (13) is

$$P(\lambda) = \lambda^6 - A_2\lambda^5 + A_1\lambda^4 - A_0\lambda^3 + A_1\lambda^2 - A_2\lambda + 1, \tag{39}$$

where the coefficients A_0, A_1, A_2 are related to the elements of the monodromy matrix $\boldsymbol{L}$ through

$$A_0 = \sum_{i<j<k} \begin{vmatrix} L_{ii} & L_{ij} & L_{ik} \\ L_{ji} & L_{jj} & L_{jk} \\ L_{ki} & L_{kj} & L_{kk} \end{vmatrix}, \qquad A_1 = \sum_{i<j} \begin{vmatrix} L_{ii} & L_{ij} \\ L_{ji} & L_{jj} \end{vmatrix}, \qquad A_2 = \mathrm{Tr}\,\boldsymbol{L}. \tag{40}$$

The three stability indices b_1, b_2, b_3 are roots of the reduced characteristic polynomial (15)

$$Q(b) = b^3 - A'_1 b^2 + A'_2 b - A'_3 = 0. \tag{41}$$

Using Eqs. (16) and (17) for $N = 3$ and by denoting $A_2 \equiv A$, $A_1 \equiv B$, $A_0 \equiv C$ for simplicity, we get

$$A = A'_1 = b_1 + b_2 + b_3, \qquad B = A'_2 + 3 = b_1 b_2 + b_1 b_3 + b_2 b_3 + 3,$$
$$C = A'_3 + 2A = b_1 b_2 b_3 + 2(b_1 + b_2 + b_3). \tag{42}$$

The discriminant of $Q(b)$ is

$$\Delta = q^2 + 4p^3, \tag{43}$$

where

$$q = -\tfrac{2}{27}A^3 + \tfrac{1}{3}A(B-3) - C + 2A, \qquad p = \tfrac{1}{3}(B-3) - \tfrac{1}{9}A^2. \tag{44}$$

The parameter space $\mathcal{S}$ where the possible stability types are defined is the three-dimensional space (A, B, C). The regions of the possible stability types in $\mathcal{S}$, are defined by the transition boundaries that correspond to the following constraints:

(i) One stability index is equal to 2. The corresponding boundary is plane p_1 shown in Fig. 5(a). The equation of this plane is obtained by putting $b = +2$ in Eq. (41) and using Eq. (42)

$$p_1: \quad C = 2(1 - A + B). \tag{45}$$

By putting $b = +2$ in Eq. (42) we get the parametric equations of plane p_1:

$$p_1: \quad A = b_1 + b_2 + 2, \qquad B = b_1 b_2 + 2(b_1 + b_2) + 3,$$
$$C = 2b_1 b_2 + 2(b_1 + b_2 + 2), \quad \text{with } b_1, b_2 \in \mathbb{R}. \tag{46}$$

(ii) One stability index is equal to -2. The boundary surface is the plane p_2 shown in Fig. 5(b). This plane is defined by the following equations:

$$p_2: \quad C = -2(1 + A + B), \tag{47}$$

or

$$p_2: \quad A = b_1 + b_2 - 2, \qquad B = b_1 b_2 - 2(b_1 + b_2) + 3,$$
$$C = -2b_1 b_2 + 2(b_1 + b_2 - 2), \quad \text{with } b_1, b_2 \in \mathbb{R}. \tag{48}$$

(iii) At least two stability indices are equal to each other. This condition is equivalent to the discriminant Δ (Eq. (43)) being equal to zero. The corresponding boundary surface p_Δ is the two-sheeted surface shown in Fig. 5(c). The upper sheet in Fig. 5(c) is denoted as p_3 and the lower sheet as p_4. The equations of the two sheets are obtained by putting $\Delta = 0$ in Eq. (43) and using Eqs. (44). So we get

$$p_3, p_4: \quad C = -\tfrac{2}{27}A^3 + \tfrac{1}{3}A(B-3) + 2A \pm 2\sqrt{-\left[\tfrac{1}{3}(B-3) - \tfrac{1}{9}A^2\right]^3}, \tag{49}$$

with " $+$" corresponding to p_3 and "$-$" to p_4. Equivalently the parametric equations of the whole surface are obtained by putting $b_1 = b_2$ to Eq. (42):

$$p_\Delta: \quad A = 2b_1 + b_3, \qquad B = b_1^2 + 2b_1 b_3 + 3, \qquad C = b_1^2 b_3 + 4b_1 + 2b_3, \quad \text{with } b_1,\ b_3 \in \mathbb{R}. \tag{50}$$

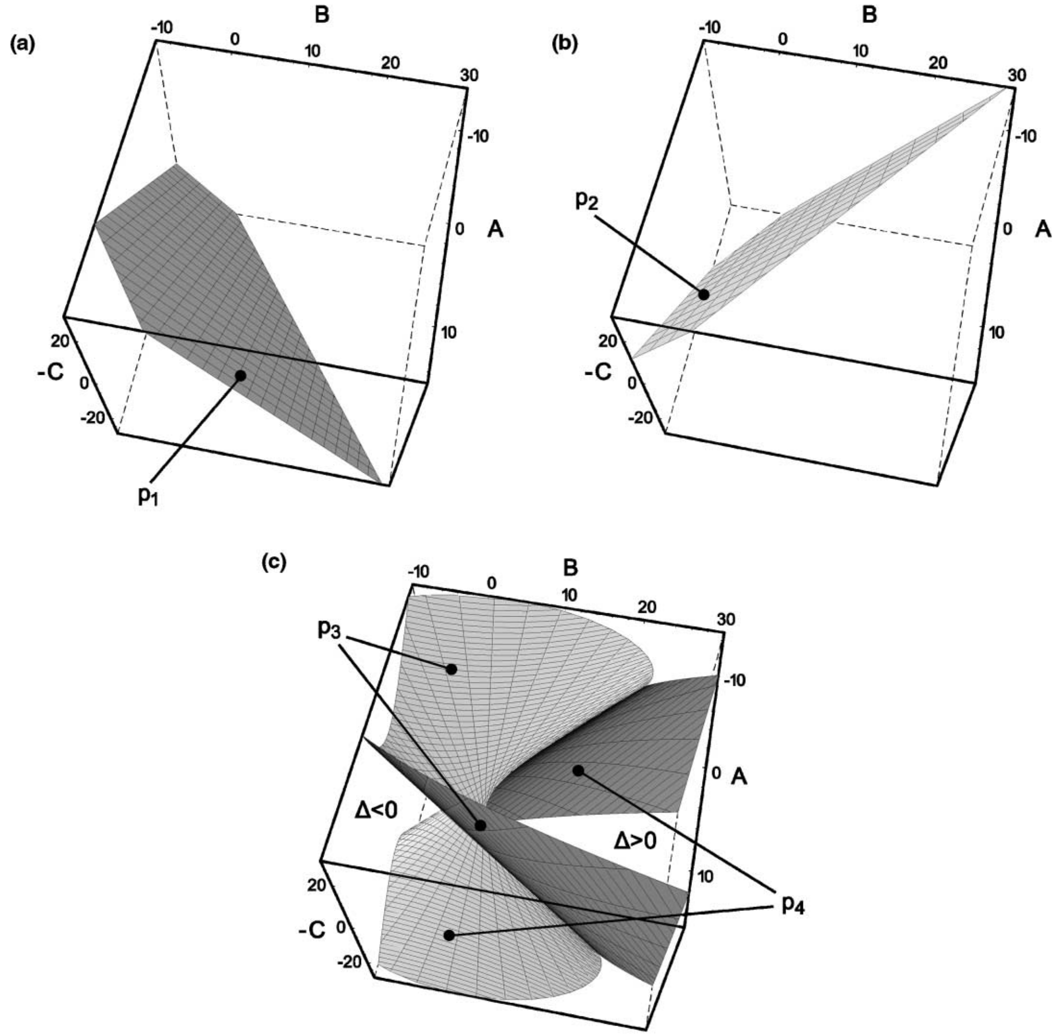

Fig. 5. The boundaries of all the possible stability regions in the parameter space $\mathcal{S}$ of a Hamiltonian system with four degrees of freedom. A, B, C are the coefficients of the corresponding characteristic polynomial. The constraint $b = +2$ corresponds to the plane p_1 (a), while the constraint $b = -2$ to the plane p_2 (b). The equality of at least two stability indices corresponds to the two-sheeted surface p_Δ in (c) composed of the p_3 and p_4 surfaces. In (c) the regions where the discriminant Δ of the reduced characteristic polynomial is positive and negative are marked.

This surface divides the parameter space $\mathcal{S}$ in the two regions shown in Fig. 5(c). In the region seen in Fig. 5(c) on the left side of p_Δ, the stability indices are three distinct real numbers ($\Delta < 0$), while in the region seen on the right side of p_Δ, we have one real stability index with the other two indices being complex conjugate numbers ($\Delta > 0$). On the surface p_Δ ($\Delta = 0$) the stability indices are real with at least two of them equal to each other. Along the curve where the two sheets p_3 and p_4 join the three stability indices are real and equal.

The equations of the intersection curves between the above surfaces are found by substituting the constraints on the stability indices, that every surface satisfies, to Eqs. (42). For example, the equation of the line of intersection of the two planes p_1 and p_2 is found by putting $b_1 = 2$ and $b_2 = -2$ in Eqs. (42). This substitution gives the parametric equation of the line:

$$p_{12}: \quad (A, B, C) = (\rho, -1, -2\rho), \quad \rho \in \mathbb{R}. \tag{51}$$

The other curves of intersection are found in a similar way. Having in mind that the two subscripts denote the number of the surfaces that intersect and $\rho \in \mathbb{R}$ we get

$$p_{13}: \quad (A, B, C) = (4 + \rho, 4\rho + 7, 6\rho + 8), \tag{52}$$

$$p_{14}: \quad (A, B, C) = (2 + 2\rho, \rho^2 + 4\rho + 3, 2\rho^2 + 4\rho + 4), \tag{53}$$

$$p_{23}: \quad (A, B, C) = (-2 + 2\rho, \rho^2 - 4\rho + 3, -2\rho^2 + 4\rho - 4), \tag{54}$$

$$p_{24}: \quad (A, B, C) = (\rho - 4, 7 - 4\rho, 6\rho - 8), \tag{55}$$

$$p_{34}: \quad (A, B, C) = \left(\rho, \tfrac{1}{3}\rho^2 + 3, \tfrac{1}{27}\rho^3 + 2\rho\right), \tag{56}$$

The points of intersection of the above curves are

$$Q_1 = (2, -1, -4), \qquad Q_2 = (-2, -1, 4), \qquad P_1 = (6, 15, 20), \qquad P_2 = (-6, 15, -20). \tag{57}$$

In particular, curves p_{12}, p_{13}, p_{23} intersect at Q_1, curves p_{12}, p_{14}, p_{24} intersect at Q_2, curves p_{13}, p_{14}, p_{34} intersect at P_1 and curves p_{23}, p_{24}, p_{34} intersect at P_2. At point Q_1 the values of the stability indices are $b_1 = 2$, $b_2 = 2$, $b_3 = -2$, while at point Q_2 they are $b_1 = 2$, $b_2 = -2$, $b_3 = -2$, at point P_1 they are $b_1 = 2$, $b_2 = 2$, $b_3 = 2$ and at point P_2 they are $b_1 = -2$, $b_2 = -2$, $b_3 = -2$.

The surface p_Δ is tangent to plane p_1 along the line p_{13}, while it intersects it along the curve p_{14}. In Fig. 6(a) the plane p_1 and the lower sheet p_4 of p_Δ are plotted. On plane p_1 the curves p_{13} and p_{14} are shown along with the point P_1 at which the two curves become tangent. The sheet p_3 is tangent to plane p_1 along the section of line p_{13} that corresponds to $\rho < 2$ in Eq. (52), and the sheet p_4 is tangent to plane p_1 along the section of p_{13} with $\rho > 2$, while $\rho = 2$ corresponds to point P_1. The sheet p_4 intersects plane p_1 along the section of curve p_{14} that corresponds to $\rho < 2$ in Eq. (53). The section of p_{14} with $\rho > 2$ corresponds to the intersection of the plane p_1 with the sheet p_3, while $\rho = 2$ corresponds to point P_1.

In a similar way, surface p_Δ is tangent to plane p_2 along the line p_{24}, while it intersects it along the curve p_{23}. In Fig. 6(b) the plane p_2 and the lower sheet p_4 of p_Δ are plotted. On plane p_2 the curves p_{23} and p_{24} are shown along with the point P_2 at which the two curves become tangent. The sheet p_3 is tangent to plane p_2 along the section of line p_{24} that corresponds to $\rho < -2$ in Eq. (55), and the sheet p_4 is tangent to plane p_2 along the section of p_{24} with $\rho > -2$, while $\rho = -2$ corresponds to point P_2. The sheet p_4 intersects plane p_2 along the section of curve p_{23} that corresponds to $\rho < -2$ in Eq. (54), while sheet p_3 intersects plane p_2 along the section of p_{24} for $\rho > -2$. The value $\rho = -2$ in Eq. (54) corresponds to point P_2.

The transition boundaries p_1, p_2 and p_Δ create 13 regions of different stability types, in agreement to Eq. (21) for $N = 3$. These regions are shown in Fig. 7. Fig. 7(a) is produced by the superposition of the three frames seen in Fig. 5. In Fig. 7(a) the regions of eight stability types are shown, in particular the regions of the stability types: $S_1U_{2,0}$, $S_2U_{1,0}$, $U_{1,0}\Delta_1$, $U_{1,2}$, $S_1U_{1,1}$, $S_2U_{0,1}$, $S_1U_{0,2}$ and $S_1\Delta_1$. In Fig. 7(b) the same portion of the parameter space $\mathcal{S}$ to the one shown in Fig. 7(a), is seen from a different point of view so that the regions of the stability types $U_{2,1}$ and $U_{0,1}\Delta_1$ are also visible. We remark that new types of instabilities are introduced in Hamiltonian systems with four degrees of freedom and in particular the types $S_1\Delta_1$, $U_{1,0}\Delta_1$ and $U_{0,1}\Delta_1$ where we have the coexistence of complex instability with stable and unstable configuration. The regions of $U_{3,0}$ and

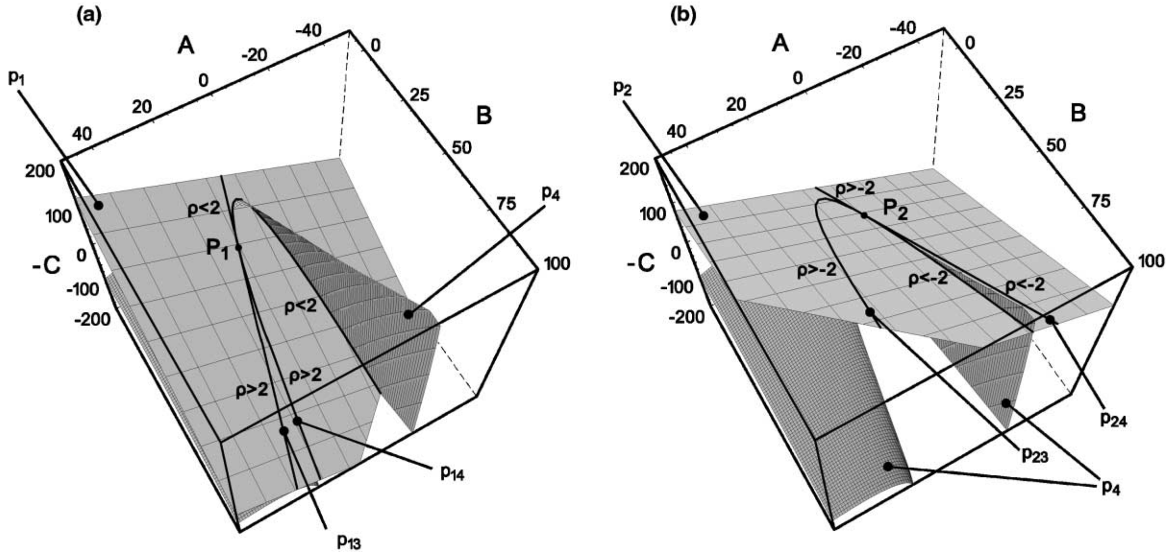

Fig. 6. Intersections and tangencies of the planes p_1 and p_2 with the surface p_Δ in the three-dimensional parameter space $\mathcal{S}$ with coordinates the coefficients A, B, C of the characteristic polynomial. (a) The plane p_1 and the lower sheet p_4 of p_Δ. The surface p_Δ is tangent to plane p_1 along the line p_{13}. In particular sheet p_3 is tangent to plane p_1 along the section of line p_{13} starting from point P_1 and towards smaller values of A ($\rho < 2$ in Eq. (52)), while at the rest part of p_{13} ($\rho > 2$ in Eq. (52)) p_4 is tangent to p_1. The surface p_Δ intersects the plane p_1 along the curve p_{14}. In particular sheet p_4 intersects p_1 along the section of curve p_{14} starting from point P_1 and towards smaller values of A ($\rho < 2$ in Eq. (53)) and sheet p_3 intersects p_1 along the rest part of p_{14} ($\rho > 2$ in Eq. (53)). (b) The plane p_2 and the lower sheet p_4 of p_Δ. The surface p_Δ is tangent to plane p_2 along the line p_{24}. In particular sheet p_3 is tangent to p_2 along the section of p_{24} starting from point P_2 and towards smaller values of A ($\rho < -2$ in Eq. (55)), while at the rest part of p_{24} ($\rho > -2$ in Eq. (55)) p_4 is tangent to p_2. The surface p_Δ intersects the plane p_2 along the curve p_{23}. In particular sheet p_4 intersects p_2 along the section of curve p_{23} starting from point P_2 and towards smaller values of A ($\rho < -2$ in Eq. (54)) and sheet p_3 intersects p_2 along the rest part of p_{23} ($\rho > -2$ in Eq. (54)).

$U_{0,3}$ are marked in Fig. 7(c). The region that corresponds to the stability type $U_{3,0}$ is located over the plane p_2 and below the p_Δ surface, while the region of $U_{0,3}$ is located below the plane p_1 and over the surface p_Δ. The boundary of the region that corresponds to the stability type S_3 is a rather complicated surface formed by the intersecting surfaces p_1, p_2 and p_Δ and it is shown in Fig. 7(d).

All the possible direct transitions between the different stability types, as well as the dimension of the corresponding transition boundary are reported in Table 3. The data of Table 3 can be found by using the information provided in Table 2 or by examining the arrangement of the various stability regions seen in Fig. 7.

From the data of Table 3 we see that certain transitions are not possible. For example, the transition $S_1U_{2,0} \rightarrow S_1U_{0,2}$ cannot happen, since the corresponding regions in $\mathcal{S}$ are not neighboring as seen in Fig. 7(a) and (b). Some transitions are possible when very specific conditions are satisfied, since they correspond to the crossing of a particular point in $\mathcal{S}$, like the transition $S_3 \rightarrow U_{3,0}$ which is performed through point P_2. On the other hand, other transitions are performed by the crossing of certain curves, like the transition $S_2U_{0,1} \rightarrow U_{1,2}$ which is performed by the crossing of line p_{12} as seen in Fig. 7(a), or by the crossing of a surface in $\mathcal{S}$, like for example the transition $S_2U_{0,1} \rightarrow S_1U_{0,2}$ since the two regions are separated by plane p_1 as seen in Fig. 7(a). The region that corresponds to the S_3 type is directly connected to all other regions. Thus, the direct transition from the stable case S_3 to any unstable type is possible.

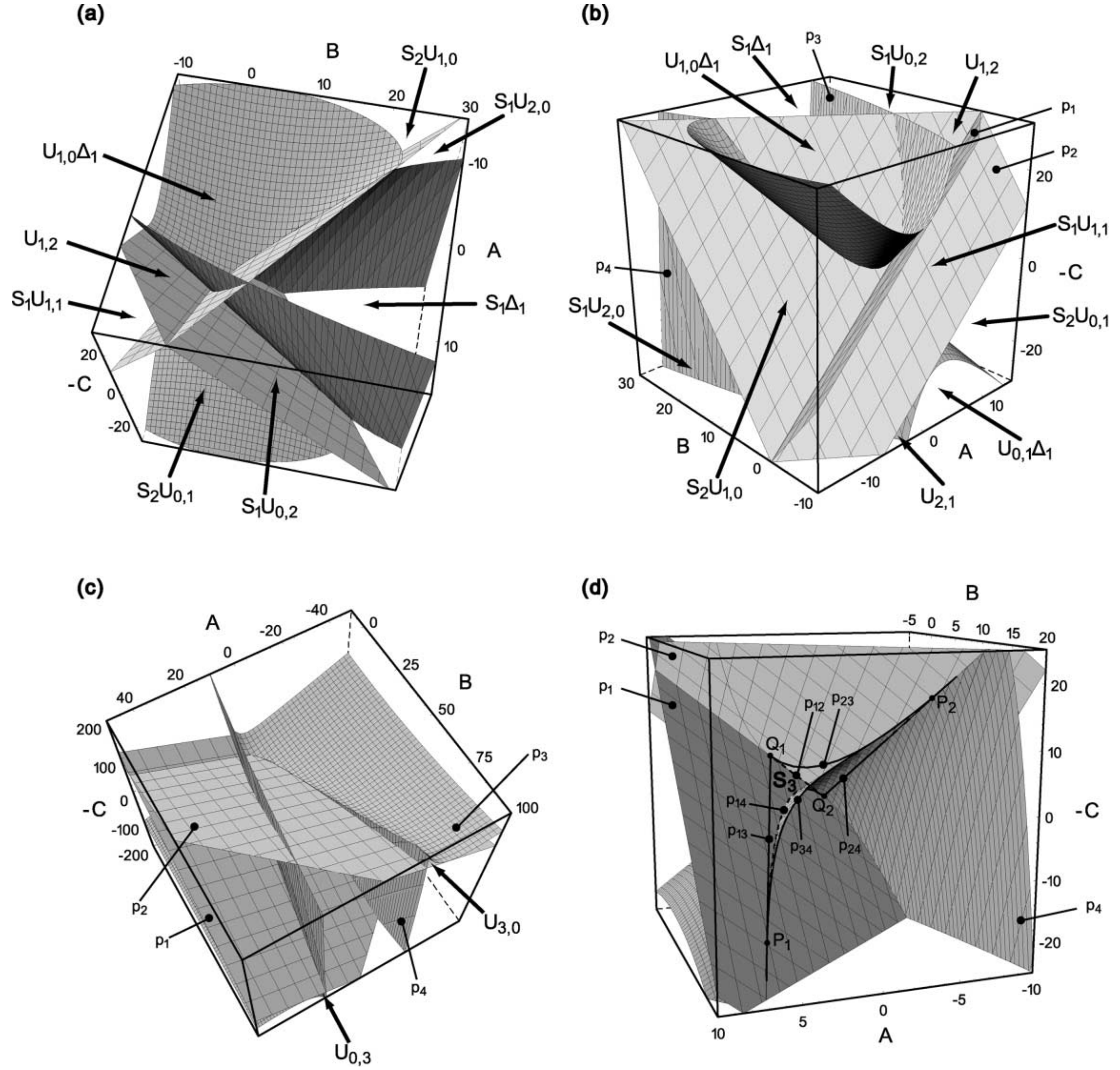

Fig. 7. The regions of the 13 possible stability types of a Hamiltonian system with four degrees of freedom, in the three-dimensional parameter space $\mathcal{S}$ with coordinates the coefficients A, B, C of the characteristic polynomial. (a) The regions of the stability types $S_1U_{2,0}$, $S_2U_{1,0}$, $U_{1,0}\Delta_1$, $U_{1,2}$, $S_1U_{1,1}$, $S_2U_{0,1}$, $S_1U_{0,2}$ and $S_1\Delta_1$. The portion of the parameter space $\mathcal{S}$ is the same to the one seen in Fig. 5, using also the same point of view. So the regions of the different stability types are produced by the superposition of the three frames of Fig. 5. (b) The same portion of the parameter space $\mathcal{S}$ to the one shown in frame (a), seen from a different point of view so that the regions of $U_{2,1}$ and $U_{0,1}\Delta_1$ are also visible. The planes p_1, p_2 and the surfaces p_3, p_4 are also marked. (c) A larger portion of the parameter space $\mathcal{S}$ compared to frames (a) and (b) (same portion to the one shown in Fig. 6 seen from the same point of view), where the spear-like regions of $U_{3,0}$ and $U_{0,3}$ are marked. The region of $U_{3,0}$ is located over the plane p_2 and below surface p_Δ. It is confined between the section of p_4 located over the plane p_2 and the section of sheet p_3 up to the line p_{24} seen in Fig. 6(b), on which p_3 is tangent to p_2. The boundary of $U_{3,0}$ on plane p_2 is the curvilinear triangular region formed by point P_2 and the sections of p_{24} and p_{23} marked by $\rho < -2$ in Fig. 6(b). The region of $U_{0,3}$ is located below the plane p_1 and over the surface p_Δ. The boundary of $U_{0,3}$ on plane p_1 is the curvilinear triangular region formed by point P_1 and the sections of p_{13} and p_{14} marked by $\rho > 2$ in Fig. 6(a). (d) The stability region S_3 is the one marked with bold lines. The region is confined by the planes p_1, p_2 and the sheets p_3, p_4. The sheet p_3 is not plotted. The boundary of S_3 on p_3 is the curvilinear triangle $P_1Q_1P_2$ with sides the sections of p_{13}, p_{23} and p_{34} marked by bold lines. The boundary of S_3 on p_4 is the curvilinear triangle $P_1Q_2P_2$ with sides the sections of p_{14}, p_{24} and p_{34} marked by bold lines. The boundary of S_3 on p_1 is the curvilinear triangle $P_1Q_1Q_2$ with sides the sections of p_{12}, p_{13} and p_{14} marked by bold lines. The boundary of S_3 on p_2 is the curvilinear triangle $P_2Q_1Q_2$ with sides the sections of p_{12}, p_{23} and p_{24} marked by bold lines.

Table 3
The possible direct transitions between the 13 different stability types of a periodic orbit of a Hamiltonian system with four degrees of freedom. The 13 stability types are listed in the first column and the first row of the table. If the direct transition between two stability types defined by the column and the row of a cell is possible then the dimension of the corresponding transition boundary is reported in the cell: 2 for a surface, 1 for a curve and 0 for a point. If a particular transition is not possible the corresponding cell contains the mark "–". The cells of the diagonal are empty since they do not represent any transition

Stability types	S_3	$S_2U_{1,0}$	$S_2U_{0,1}$	$S_1U_{2,0}$	$S_1U_{1,1}$	$S_1U_{0,2}$	$S_1\Delta_1$	$U_{3,0}$	$U_{2,1}$	$U_{1,2}$	$U_{0,3}$	$U_{1,0}\Delta_1$	$U_{0,1}\Delta_1$
S_3		2	2	1	1	1	2	0	0	0	0	1	1
$S_2U_{1,0}$	2		1	2	2	0	1	1	1	1	–	2	0
$S_2U_{0,1}$	2	1		0	2	2	1	–	1	1	1	0	2
$S_1U_{2,0}$	1	2	0		1	–	2	2	2	–	–	1	1
$S_1U_{1,1}$	1	2	2	1		1	0	–	2	2	–	1	1
$S_1U_{0,2}$	1	0	2	–	1		2	–	–	2	2	1	1
$S_1\Delta_1$	2	1	1	2	0	2		1	1	1	1	2	2
$U_{3,0}$	0	1	–	2	–	–	1		–	–	–	2	–
$U_{2,1}$	0	1	1	2	2	–	1	–		–	–	–	2
$U_{1,2}$	0	1	1	–	2	2	1	–	–		–	2	–
$U_{0,3}$	0	–	1	–	–	2	1	–	–	–		–	2
$U_{1,0}\Delta_1$	1	2	0	1	1	1	2	2	–	2	–		–
$U_{0,1}\Delta_1$	1	0	2	1	1	1	2	–	2	–	2	–	

5. Summary

We considered the problem of the stability of periodic orbits of autonomous Hamiltonian systems with $N + 1$ degrees of freedom or equivalently of $2N$-dimensional symplectic maps, where N is an integer with $N \geq 1$. The stability of a periodic orbit is defined by the eigenvalues of the corresponding monodromy matrix, which are given as roots of the characteristic polynomial (13), or equivalently by the values of the stability indices (14) which are provided as roots of the reduced characteristic polynomial (15). The introduction of the stability indices simplifies the mathematical formalism, since the order of the reduced characteristic polynomial is half the order of the characteristic polynomial. The coefficients of the reduced characteristic polynomial are related to the stability indices and to the coefficients of the characteristic polynomial through Eqs. (16) and (17), respectively.

The results of our study can be summarized as follows:

(i) We introduce a new terminology for the different stability types that a periodic orbit of a Hamiltonian system with $N+1$ degrees can exhibit. That terminology is perfectly suited for systems with many degrees of freedom, since it provides in a clear way the configuration of the eigenvalues of the monodromy matrix on the complex plane. The general form of a stability type is

$$S_n U_m \Delta_l, \quad \text{with } n + m + 2l = N,$$

which means that n couples of eigenvalues are on the unit circle, m couples are on the real axis and l quartets are on the complex plane but not on the unit circle and not on the real axis.

(ii) All the possible U_m types are not identical, since a pair of negative real eigenvalues cannot become positive under a continuous change of a parameter of the system. So, by taking into account the different configurations of the eigenvalues that correspond to the U_m case, the introduced notation becomes

$$S_n U_{m_1,m_2} \Delta_l, \quad \text{with } m_1 + m_2 = m,$$

which means that there exist m_1 real negative pairs of eigenvalues and m_2 real positive pairs of eigenvalues.

(iii) The stability type of a periodic orbit is represented by a point in the N-dimensional parameter space $\mathcal{S}$ with coordinates the coefficients of the characteristic polynomial. So different stability types correspond to different regions of space $\mathcal{S}$. The number of the different regions is given by Eq. (21).

(iv) We register all the possible direct transitions between different stability types. The dimension D of the hypersurface in $\mathcal{S}$, which corresponds to a certain transition is an indicator of how probable this transition is. The constraints on the stability indices that the transition

$$S_n U_m \Delta_l \to S_{n+\delta n} U_{m+\delta m} \Delta_{l+\delta l}$$

introduces and the dimension D of the corresponding transition hypersurface in $\mathcal{S}$ are provided in Table 1. In this table the different arrangements of the eigenvalues for the U_m and $U_{m+\delta m}$ types have not been taken into account. So the transitions registered in Table 1 can happen in the sense that there exists at least one configuration of the eigenvalues, compatible to the U_m and $U_{m+\delta m}$ types, that allows the transition to occur. Taking into account the possible different arrangements of the eigenvalues for the U_m type and using the notation

$$S_n U_{m_1,m_2} \Delta_l \equiv [n, m_1, m_2, l],$$

the possible direct transitions and the dimension of the corresponding transition hypersurface in $\mathcal{S}$, are found by using repetitively the transition cases listed in Table 2, without using the same pair or quartet of eigenvalues twice. An explicit algorithm for determining whether a transition is possible and if so, finding the dimension of the corresponding hypersurface in $\mathcal{S}$ is provided in Section 3.3.2.

(v) We applied the new terminology of the stability types to the well-known cases of Hamiltonian systems with two and three degrees of freedom, referring also to the various direct transitions between different stability types as they arise from Tables 1 and 2. We also studied in detail the three-dimensional parameter space $\mathcal{S}$ of a Hamiltonian system with four degrees of freedom or equivalently of a six-dimensional symplectic map. By providing the equations of the boundary surfaces of the different stability regions, we define the regions in $\mathcal{S}$ that correspond to all the possible instabilities not limiting ourselves in studying only the stable case S_3. The arrangement of the various regions define the possible direct transitions between different stability types, in agreement to the information provided in Tables 1 and 2. All the direct transitions as well as the dimension of the corresponding transition boundary are reported in Table 3.

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